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Sustainable ecosystem management using optimal control theory: Part 1 (deterministic systems)

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Abstract

The concept of sustainability, an abstract one by its nature, has been given a mathematical representation through the use of Fisher information as a measure. It is used to propose the sustainability hypotheses for dynamical systems, which has paved the way to achieve sustainable development through externally enforced control schemes. For natural systems, this refers to the task of ecosystem management, which is complicated due the lack of clear objectives. This work attempts to incorporate the idea of sustainability in ecosystem management. The natural regulation of ecosystems suggests two possible control options, top-down control and bottom-up control. A comparison of these two control philosophies is made on generic food chain models using the objectives derived from the sustainability hypotheses. Optimal control theory is used to derive the control profiles to handle the complex nature of the models and the objectives. The results indicate a strong relationship between the hypotheses and the dynamic behavior of the models, supporting the use of Fisher information as a measure. As regards to ecosystem management, it has been observed that top-down control is more aggressive but can result in instability, while bottom-up control is guaranteed to give a stable and improved dynamic response. The results also indicate that bottom-up control is a better option to affect shifts in the dynamic regimes of a system, which may be required to recover the system from a natural disaster like the hurricane Katrina.

Keywords: Sustainability; Fisher information; Ecosystem management; Optimal control; Food chain models

1. Introduction

Sustainability is a relatively new concept in the field of ecology. Sustainable development, a multifaceted approach to manage the environmental, economic, and social resources, calls for the consideration of the long-term effects in all the decisions relevant to the society as a whole. Being embodied in a multidisciplinary environment, a suitable mathematical representation, or a measure, of sustainability is essential for the successful communication amongst various fields encompassed by the concept. To this effect, Cabezas and Fath (2002) have proposed Fisher information (FI) as a sustainability measure for dynamic systems, and have formulated the sustainability hypotheses, with particular focus on the natural ecosystems. These hypotheses help rank various dynamic systems and decide the nature of evolution, using sustainability as a criteria. As regards to natural evolution, there is not much that humans can do about. However, when it comes to the evolution caused by human interventions (e.g. ecosystem management), such a criteria can be helpful in deciding the right course of action.

Ecosystem management is much more complicated than the management of engineering systems due to the complexities of the natural systems. Not only are these systems poorly understood, but also the objectives to be achieved are quite obscure. This is where the multi-disciplinary concept of sustainability can make an invaluable contribution. Binding the ecosystem properties of disparate temporal and spatial scales together, it becomes an effective tool for ecosystem management. The work by Cabezas and Fath (2002) has made the quantification of sustainability possible, paving the way to implement this

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abstract concept in management related decision making. This work, if broadly defined, concerns with the application of the sustainability concept in the management of ecosystems (such as lakes), by manipulating species populations.

A key to effectively regulate any ecosystem is to first understand the natural regulation of the system. For this, the scientists have proposed two different control philosophies: top-down control (Carpenter et al., 1985) and bottom-up control (McQueen et al., 1986). These natural regulation paths can be used to advantage when trying to impose external control for ecosystem management. There has been an intense debate over the validity and relative importance of both the philosophies, and the general consensus to emerge is that both the regulation paths are dominant at different levels of the ecosystem represented as a food chain (Brett and Goldman, 1997). This work performs a relative assessment of these two control philosophies.

However, given the complexities of the natural systems and objective (sustainability), a heuristics based derivation of the control profiles is not justified. The advanced control theory has been developed to tackle such problems, and optimal control theory has been at the forefront due to some of its obvious advantages. This work, therefore, uses the optimal control theory to formulate the control profiles. One of the necessities of the optimal control theory is a model of the system. This work models the species interactions in an ecosystem using a three species predator–prey model, a class of the general food web models.

To summarize, this work compares the top-down and bottom-up control philosophies, derived using the optimal control theory, for a population model in an ecosystem, using sustainability as the objective, which is quantified by FI. Fig. 1 shows the schematic explaining the relative contribution from each of the topics mentioned here. This paper presents the first part of the work dealing with the deterministic models. The second part deals with the models having uncertainties, and compares the results with those for the deterministic models.

The article is organized as follows. The next section reviews the theory behind the work and Section 3 gives the problem specific details. Sections 4 and 5 report and discuss the results for a three species predator–prey model. The article ends with comments on the computational aspects in Section 6 and conclusions in Section 7.

2. Theoretical basics

2.1. Sustainability and Fisher information

Sustainable development is defined as "the development that meets the needs of the present without compromising the ability of the future generations to meet their own needs" (Tomlinson, 1987). In its simplest terms, it calls for the consideration of the long-term effects, benefits and drawbacks in all the decisions relevant to the society as a whole. Since its formalization, this new concept has become increasingly important and popular, attracting active research. Although the concept has been universally recognized to be of paramount importance, it is essential to have a quantifying measure to use it in the field of ecosystem management.

Cabezas and Fath (2002) have proposed to use information theory in ecology to derive a measure for the

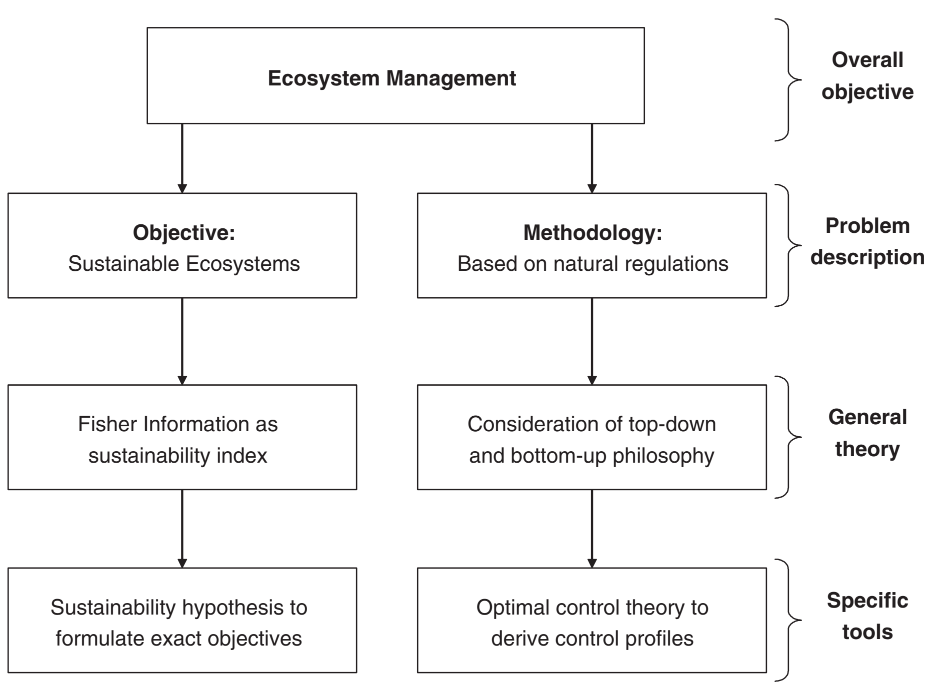

Fig. 1. Integration of sustainability and ecosystem management: theory and tools.

sustainability of a system, the hypothesis being based on the argument that information is a fundamental quantity of any system, irrespective of the discipline (Frieden, 1998). The previous applications of information theory in ecology include: using Shannon information (Shannon and Weaver, 1949) as an index of biodiversity; using the entropy of information to investigate the evolutionary processes (Brooks and Dodson, 1965); using information about the energy pathways to quantify the interdependencies and diversity in a food web (Rutledge et al., 1976); measuring the distance of a system from the thermodynamic equilibrium (based on exergy) (Mejer and Jørgensen, 1979) and developing the concept of ascendancy (Ulanowicz, 1986). Cabezas and Fath use FI as the quantity for their hypothesis.

Fisher information (FI), introduced by Ronald Fisher (Fisher, 1922), is a statistical measure of indeterminacy. One of its interpretations, relevant for this work, is as a measure of the state of order or organization of a system or phenomenon (Frieden, 1998). Fisher information, I, for one variable is given as (Cabezas and Fath, 2002)

$$I = \int \frac{1}{p(x)} \left(\frac{\mathrm{d}p(x)}{\mathrm{d}x} \right)^2 \mathrm{d}x, \tag{1}$$

where, p is the probability density function (*pdf*) of variable x. This definition can be extended to a system of n variables. When these n variables constitute the state variable vector of a system, it gives FI of that system. FI, being a local property, dependent on the derivative of the density function of the state variable vector, is sensitive to the perturbations that affect the density function, and therefore, can be used as an indicator of the organization of the system. Here, organization refers to the distribution of the states in which the system exists. A system with many equally probable states is disorganized, while a system with a few preferred states is better organized. For a highly disorganized system, the lack of predictability, due to the nearly uniform probability distribution of its states, results in a low value of FI. On the contrary, a highly organized system will have a high value of FI. The central argument in the sustainability hypothesis (Cabezas and Fath, 2002) is that the stability (static or dynamic) of a system is sufficient (but not necessary) for the sustainability of the state of the system. A statically or dynamically stable system, due to one or a few preferred states, will have a high value of FI. Thus FI, being an indicator of the system stability, is also an indicator of the sustainability of the state of the system. For the stability of an ecosystem (static or dynamic), it is important that the system is not losing or gaining species, affecting the system dimensionality and hence the value of FI. The sustainability hypothesis, therefore, states that: the time-averaged FI of a system in a persistent regime does not change with time. Any change in the regime will manifest itself through a corresponding change in FI value (Cabezas and Fath, 2002).

Two additional corollaries to this hypothesis, stated by Cabezas and Fath (2002), are: (1) if FI of a system is increasing with time, then the system is maintaining a state of organization, and (2) if FI of a system is decreasing with time, then the system is losing its state of organization. These corollaries are based on the correlation between the system order and FI, and they give an idea about the quality of change, if the system is changing its state. An extensive review of FI and the sustainability hypotheses can be found in Cabezas and Fath (2002) and Fath et al. (2003).

The sustainability hypotheses provide the theoretical basis for the work presented in this paper. From an ecosystem management perspective, two different objectives can be formulated based on these hypotheses:

- Maximization of time averaged FI. This objective is based on the idea that higher FI is a consequence of a more organized system. Hence, the objective attempts to push the system from its current state into a state that is more sustainable in the mathematical sense of FI. It may demand rapid changes though.
- Minimization of the FI variance over time. This objective is a direct consequence of the sustainability hypothesis. Minimizing the FI variance ensures the constancy of regime. The objective thus aims to maintain the system close to its current state.

These different objectives are compared for different control philosophies. These control philosophies are explained in the following section.

2.2. Ecosystem management philosophies

This section gives a brief overview of the different control philosophies that have been proposed for the natural regulation of an ecosystem. It will help in deciding the external control philosophies.

The earliest attempt to systematically understand the controlling effects of a natural ecosystem dates back to Hairston et al. (1960). This has since encouraged more and more research in understanding the natural regulation of the ecosystems, and it has led to the concept of trophic cascade hypothesis (Carpenter et al., 1985; Paine, 1980). For an aquatic ecosystem, it proposes that the predator–prey interactions are transmitted through the food webs to cause variance in the phytoplankton biomass and production, at constant nutrient load (Carpenter et al., 1985), and that the responses are nonlinearly related to the strength of the interactions among the adjacent trophic levels. Another possible regulatory effect in a food web is that of the available resources on the higher level species, e.g. nutrients support the phytoplankton biomass in an aquatic food web, which in turn affects the top level species that feed on it.

From the ecosystem management point of view, this has led to the formulation of two different control philosophies: top-down control and bottom-up control.

Top-down control (also called as consumer control) refers to the control of the ecosystem through the top level predators. Bottom-up control (also called as resource control) refers to the control of the ecosystem via the available resources (e.g. nutrients). A debate has been going on over the validity of the individual control philosophies (Carpenter and Kitchell, 1988; Townsend, 1988; McQueen et al., 1989), and also, over the relative importance of each of those in a food web (Lynch and Shapiro, 1981; Vanni and Temte, 1990; Rosemond et al., 1993). The recent opinion, based on some of the published results, is that both the effects are apparent in a food chain, and the relative importance of the two depends on the length of the food chain and the position of a particular species in the food chain (Brett and Goldman, 1997; McQueen et al., 1986). Thus, top-down control is more prominent in species at the top of the food chain, while the lower level species are more strongly under bottom-up control. Most of these results are based on experimental manipulations, followed by observations over a long period of time.

The existence of these two natural regulation paths in ecosystems provides two different avenues to exercise the external control of these systems. Thus, regulation by controlling the top predator and by controlling the lowest level resources are the two options explored and compared in this work.

2.3. Optimal control theory

Control theory aims to derive a time dependent profile of a particular system parameter, called the control variable, such that a specific objective is optimized over the considered time horizon. The development of the control theory has primarily been motivated to solve the performance problems of engineering systems e.g. mechanical, electrical, chemical etc. Environmental problems, such as ecosystem management, also offer an exciting avenue for implementing some of the advanced control strategies. Some examples of the use of control theory in ecosystem management include: population management through harvesting (Kolosov, 1997; Kolosov and Sharov, 1993; Sivert and Smith, 1977), lake water quality management (Ludwig et al., 2003) and forest fire management (Richards et al., 1999; Anderson, 1994). Some other applications of control theory in natural system management include Carlson et al. (1991) and Chukwu (2001). The application of optimization to make time independent decisions, such as those in Hof and Bevers (2002) and Mees and Strauss (1992), is easier than using a time dependent control. However, the environmental systems are constantly evolving. Implementing time independent decisions, overlooking such dynamics, might be sub-optimal. It may achieve the short-term goals but hamper the long-term objectives. For example, limiting the nutrient or pollutant input into a lake at a constant level, without giving proper credence to the environmental cycles and the aquatic life cycles in the lake, can affect the species diversity and their life expectancy. Such effects are not evident immediately and manifest themselves only over a longer time period. Sustainability demands the consideration of such long-term objectives. Hence, to achieve sustainable systems, the time dependent nature of the system parameters needs to be considered. Moreover, there are multiple parameters in nature, which are partially or completely under human control. In such cases, a rigorous mathematical analysis needs to replace the decision making based on experience and logic. The use of the advanced control strategies, therefore, might not just be an option, but rather a necessity.

Optimal control is at the forefront of the advanced control strategies. Some advantages of optimal control over the other advanced control strategies are: it does not make any assumption about the form of the control law, it theoretically gives the best control strategy for the given objective function, and it can theoretically handle any type of system. Owing to the complexity of the natural systems and the sustainability based objectives, this work uses the theory of optimal control to derive the top-down and bottom-up control profiles for the population models in an ecosystem.

The theory presents three possible methodologies to derive the optimal control law: dynamic programming (Hamilton–Jacobi–Bellman equation), calculus of variation (Euler–Lagrange equation) and Pontryagin's maximum principle (Kirk, 1970; Diwekar, 2003). In this work, Pontryagin's maximum principle has been used. A detailed explanation of the theory is beyond the scope of this article, and only a brief overview of the final equations is given here. The interested readers are referred to Kirk (1970).

Consider a system represented by the following set of differential equations, called the state equations in the language of control theory

$$\dot{x} = f(x, u, t), \tag{2}$$

where, x is the state variable vector ($x(t) \in R^n$). The initial condition of the state variable vector is $x(t_0) = x_0$, while the final conditions is $x(T)$. u is the control variable vector ($u(t) \in R^m$). In optimal control, there is a time dependent performance index. It is represented here as

$$J(t_0) = \int_{t_0}^{T} F(x(t), u(t), t)\,\mathrm{d}t, \tag{3}$$

where, F is the function to be optimized over the time interval $[t_0, T]$. The optimal control theory converts this integral objective into a Hamiltonian (calculated at each time step), which is defined as

$$H(x, u, t) = F(x, u, t) + \lambda' f(x, u, t), \tag{4}$$

where, λ is a set of the costate or adjoint variables (R^n) (λ' represents the matrix transpose). The optimal control law is then given by the solution of the following set of equations:

State equations:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f, \quad t \geqslant t_0. \tag{5}$$

Adjoint equations:

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f'}{\partial x}\lambda + \frac{\partial F}{\partial x}, \qquad t \leqslant T. \tag{6}$$

Optimality conditions:

$$0 = \frac{\partial H}{\partial u} = \frac{\partial F}{\partial u} + \frac{\partial f'}{\partial u}\lambda. \tag{7}$$

This is a set of $2n$ ordinary differential equations (state and adjoint equations) and m algebraic equations (optimality or stationarity condition), to be solved as a boundary value problem. The state variables are knows at the initial time t_0 (x_0), while the adjoint variables are known at the final time T ($\lambda(T)$). The boundary values of the adjoint variables depend on the problem specification (Kirk, 1970). The control trajectory obtained using the optimality condition is optimal for the considered objective function and the starting conditions.

The application of optimal control theory necessitates the knowledge of three aspects, the objective, the control variable and the model of the given system. The objectives are given by the sustainability hypothesis, while the control variable is decided by the different control philosophies. The model of the system is explained in the next section.

2.4. Predator–prey model

The model used is this work to represent the species interactions is the predator–prey model, derived from the more general class of Lotka–Volterra-type models. These models give a simplistic mathematical representation of the observed dynamics of the various species populations in nature. In many applications, particularly those related to the aquatic systems, three level food chain models are often a good enough representation of the animal community (Ryan Gwaltney et al., 2003). This work uses the Rosenzweig–MacArthur model, which is frequently used in theoretical ecology (Abrams and Roth, 1994; De Feo and Rinaldi, 1997; Gragnani et al., 1998). The model is given by the following set of differential equations:

$$f_1 = \frac{dx_1}{dt} = x_1\left[r\left(1 - \frac{x_1}{K}\right) - \frac{a_2x_2}{b_2 + x_1}\right], \tag{8}$$

$$f_2 = \frac{dx_2}{dt} = x_2\left[e_2\frac{a_2x_1}{b_2 + x_1} - \frac{a_3x_3}{b_3 + x_2} - d_2\right], \tag{9}$$

$$f_3 = \frac{dx_3}{dt} = x_3\left[e_3\frac{a_3x_2}{b_3 + x_2} - d_3\right], \tag{10}$$

where, x_1, x_2 and x_3 are the population variables of three different species in the food chain, in the ascending order of the position in the chain. These species are referred to as the prey (x_1), predator (x_2) and super-predator (x_3) in the subsequent text. r and K are the prey growth rate and the prey carrying capacity, respectively, and a_i, b_i, e_i and d_i, $i = 2, 3$, are the maximum predation rate, half saturation constant, efficiency, and death rate of the predator ($i = 2$) and the super-predator ($i = 3$). $x_i(0)$ is the population of species i at the starting time. The model parameters for a dynamically stable system are given in Table 1. For these parameter values, this model shows cyclic variations in the species populations (biomass), and the average population of each species remains steady.

Table 1
Tri-trophic food chain model: stable parameter set

Prey	Predator	Super-predator
$x_1(0) = 100$	$x_2(0) = 75$	$x_3(0) = 150$
$r = 1.2$	$a_2 = 2.0$	$a_3 = 0.1$
$K = 710$	$b_2 = 200$	$b_3 = 250$
	$e_2 = 1.0$	$e_3 = 1.12$
	$d_2 = 1.0$	$d_3 = 0.04$

3. Optimal control problem specifications

Based on the background theory explained in the last section, this section gives the formulations used in this work.

The objectives, as mentioned in Section 2.1, are: maximization of time averaged FI and minimization of the FI variance over time. The definition of FI given by Eq. (1) is in terms of the state variable vector x. For dynamic systems, there is a one to one correspondence between the system evolution (states) and time. Using this relationship and the chain rule of differentiation, the system *pdf* and FI can be defined in terms of time. Time averaged FI for a system with n species is thus given by

$$I_t = \frac{1}{T_c}\int_0^{T_c}\left(\frac{a(t)^2}{v(t)^4}\right)dt, \tag{11}$$

where T_c is the cycle time of the system and

$$v(t) = \sqrt{\sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)^2}, \tag{12}$$

$$a(t) = \frac{1}{v(t)}\left[\sum_{i=1}^{n}\frac{dx_i}{dt}\frac{d^2x_i}{dt^2}\right], \tag{13}$$

$v(t)$ and $a(t)$ are called the velocity and acceleration terms of the ecosystem, respectively. Please refer to Fath et al. (2003) for a formal derivation of Eq. (11).

The objectives are given as

- Maximization of FI:

$$J = \mathrm{Max}\,\frac{1}{T}\int_0^{T}\left(\frac{a(t)^2}{v(t)^4}\right)dt. \tag{14}$$

- Minimization of the FI variance:

$$J = \text{Min} \int_0^T (I_t - I_{constant})^2 \, dt. \tag{15}$$

Here, T is the total time horizon under consideration. I_t, given by Eq. (11), is time averaged FI for one system cycle, and $I_{constant}$ is the constant around which the FI variation is to be minimized.

The top-down and bottom-up control philosophies are compared by performing separate analyses using x_3 and x_1, the super-predator and the prey, as the control variable, respectively. x_3 is controlled by manipulating the mortality rate d_3 of the super-predator, and x_1 is controlled by manipulating the parameter K, which represents the prey carrying capacity of the system. The control by manipulating K does not strictly represent the bottom-up approach, which proposes control by nutrient addition. However, nutrient addition affects the prey carrying capacity of the system (Abrams, 1993). The results are, therefore, expected to give the impact of the manipulation in the lower level of the food chain on the upper level in the chain.

The standard procedure of Pontryagin's Maximum principle is followed to develop the control equations. Given below are the general equation forms for the objective of FI maximization. The state equations are given by Eqs. (8)–(10), while the Hamiltonian is given by

$$H(x, u, t) = F + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3. \tag{16}$$

In this case, F is given as

$$F = \frac{1}{T}\left(\frac{R_2^2}{R_1^3}\right), \tag{17}$$

where, R_1 and R_2 are given by Eqs. (18) and (19) as follows:

$$R_1 = \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 = (f_1)^2 + (f_2)^2 + (f_3)^2, \tag{18}$$

$$R_2 = \frac{dx_1}{dt}\frac{d^2x_1}{dt^2} + \frac{dx_2}{dt}\frac{d^2x_2}{dt^2} + \frac{dx_3}{dt}\frac{d^2x_3}{dt^2} = f_1\bar{f_1} + f_2\bar{f_2} + f_3\bar{f_3}. \tag{19}$$

Here, $\bar{f_1}$, $\bar{f_2}$ and $\bar{f_3}$ are the second derivatives of x_1, x_2 and x_3 with respect to time, respectively. The adjoint equations for the tri-trophic food chain model are given as

$$-\dot{\lambda}_i = \frac{\partial H}{\partial x_i}, \quad t \leqslant T \quad \text{and} \quad i = 1, 2, 3, \tag{20}$$

where

$$\frac{\partial H}{\partial x_i} = \lambda_1 \frac{\partial f_1}{\partial x_i} + \lambda_2 \frac{\partial f_2}{\partial x_i} + \lambda_3 \frac{\partial f_3}{\partial x_i} + \frac{\partial F}{\partial x_i}, \tag{21}$$

$$\frac{\partial F}{\partial x_i} = \frac{1}{T}\frac{R_2}{R_1^3}\left[2\frac{\partial R_2}{\partial x_i} - 3\frac{R_2}{R_1}\frac{\partial R_1}{\partial x_i}\right], \tag{22}$$

$$\frac{\partial R_1}{\partial x_i} = 2f_1\frac{\partial f_1}{\partial x_i} + 2f_2\frac{\partial f_2}{\partial x_i} + 2f_3\frac{\partial f_3}{\partial x_i}, \tag{23}$$

$$\frac{\partial R_2}{\partial x_i} = f_1\frac{\partial \bar{f_1}}{\partial x_i} + \bar{f_1}\frac{\partial f_1}{\partial x_i} + f_2\frac{\partial \bar{f_2}}{\partial x_i} + \bar{f_2}\frac{\partial f_2}{\partial x_i} + f_3\frac{\partial \bar{f_3}}{\partial x_i} + \bar{f_3}\frac{\partial f_3}{\partial x_i}. \tag{24}$$

The optimality/stationarity condition is given by Eq. (25).

$$\frac{\partial H}{\partial u} = 0, \tag{25}$$

where,

$$\frac{\partial H}{\partial u} = \lambda_1 \frac{\partial f_1}{\partial u} + \lambda_2 \frac{\partial f_2}{\partial u} + \lambda_3 \frac{\partial f_3}{\partial u} + \frac{\partial F}{\partial u}, \tag{26}$$

$$\frac{\partial F}{\partial u} = \frac{1}{T}\frac{R_2}{R_1^3}\left[2\frac{\partial R_2}{\partial u} - 3\frac{R_2}{R_1}\frac{\partial R_1}{\partial u}\right], \tag{27}$$

$$\frac{\partial R_1}{\partial u} = 2f_1\frac{\partial f_1}{\partial u} + 2f_2\frac{\partial f_2}{\partial u} + 2f_3\frac{\partial f_3}{\partial u}, \tag{28}$$

$$\frac{\partial R_2}{\partial u} = f_1\frac{\partial \bar{f_1}}{\partial u} + \bar{f_1}\frac{\partial f_1}{\partial u} + f_2\frac{\partial \bar{f_2}}{\partial u} + \bar{f_2}\frac{\partial f_2}{\partial u} + f_3\frac{\partial \bar{f_3}}{\partial u} + \bar{f_3}\frac{\partial f_3}{\partial u}. \tag{29}$$

Here, u can either be K or d_3, depending on the control philosophy.

Eq. (25) gives an implicit relationship for the control variable u as a function of the states x_1, x_2 and x_3 and the model parameters. Owing to the complexity of the equation, an explicit relationship for u cannot be derived. All the contributing equations are reported in the Appendix. The starting conditions for the system (populations at the start), $x_1(0)$, $x_2(0)$ and $x_3(0)$, are known, and since the final states are free, the adjoint variables at the final time are zero, i.e. $\lambda_1(T) = 0$, $\lambda_2(T) = 0$ and $\lambda_3(T) = 0$. The final state of the system is not constrained, and the objective does not contain a final time function. The time horizon for the control problem is considered to be large enough so that the control law is only state dependent (Kirk, 1970). The derivation of the optimal control law requires the solution of the two point boundary value problem, which, for this highly complex differential-algebraic system of equations, is a cumbersome task. The numerical technique of the steepest ascent of Hamiltonian is used to solve the boundary value problem (Kirk, 1970; Diwekar, 1996). The technique solves the problem as an optimization problem by discretizing the solution horizon, using the control variable at each time instant as a decision variable, and trying to satisfy the optimality condition at each time point within a tolerance limit.

The next section gives the simulation results for the three species predator–prey model.

4. Simulation results and discussion

The analysis compares both control options and both the objectives on the tri-trophic food chain model given by Eqs. (8)–(10). Moreover, the uncontrolled model is considered to have undesirable dynamics. It simulates the situations when the ecosystem needs external intervention to avoid imbalance. The objective is to recover the system from the disturbance in a sustainable manner, i.e. achieving dynamic stability. The analysis assesses the ability of the different sustainability based objectives and control philosophies to perform this task. Three such cases are considered:

- Case A: Excessive prey and predator variations—populations x_1 and x_2 vary excessively.
- Case B: Super-predator extinction—population x_3 is going extinct.
- Case C: Super-predator explosion—population x_3 is exploding.

These cases are simulated by modifying the model parameters reported in Table 1. The changes are mentioned in the discussion of the respective cases. When K or d_3 is the control variable, the reported value is the starting guess for the steepest ascent algorithm. The control variables are constrained to avoid numerical problems. The maximum principle cannot take care of the bounds on the control variables. However, since the work uses the steepest ascent of Hamiltonian algorithm, which approximates the solution of maximum principle, it is possible to include constraints on the control variables. The bounds are also important from the implementation perspective. Most often, the control actions by humans are bounded by the natural processes. For example, the mortality rate will be bounded by the natural reproduction rate of the super-predators, and the natural input of the nutrients to the lake will limit the lowest possible nutrient addition. Hence, the incorporation of such bounds after considering the physical limitations will actually help in giving the solutions that are optimal under the given restrictions.

The model is first simulated for the uncontrolled case and then subjected to the different control options. The constraints on the control variables are: $0.03 \leqslant d_3 \leqslant 0.05$ for top-down control and $500 \leqslant K \leqslant 900$ for bottom-up control. The simulation results for the three cases are discussed in the following sections.

4.1. Case A: excessive prey and predator variations

The model parameters used to simulate this case are reported in Table 2. Although the uncontrolled system is dynamically stable, the increased variation in the prey and predator populations is undesirable. This is because when the population size is small during the cycle, they are in a greater danger of becoming extinct due to unpredicted events such as natural disasters, external species invasion etc.

The numerical values for the results are reported in Table 3, which illustrate that all the control options achieve the desired objectives to different extents. The predator–prey dynamics for these results are shown in Figs. 2 and 3. The results for top-down control, plotted in Fig. 2, show

Table 2
Tri-trophic food chain model: parameter set for case A

Prey	Predator	Super-predator
$x_1(0) = 100$	$x_2(0) = 75$	$x_3(0) = 150$
$r = 1.2$	$a_2 = 2.0$	$a_3 = 0.1$
$K = 710$	$b_2 = 235.3$	$b_3 = 250$
	$e_2 = 1.35$	$e_3 = 1.29$
	$d_2 = 1.0$	$d_3 = 0.04$

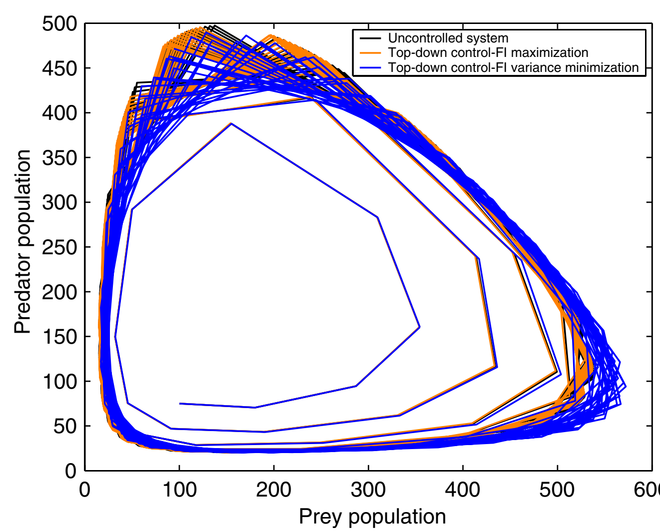

Fig. 2. Predator–prey population dynamics for case A: top-down control.

Table 3
Results for case A: excessive prey and predator population variation

Type of analysis	Top-down control		Bottom-up control	
	FI	FI standard deviation	FI	FI standard deviation
Uncontrolled model	4.68×10^{-5}	9.29×10^{-6}	4.68×10^{-5}	9.29×10^{-6}
FI maximization	4.69×10^{-5}	9.59×10^{-6}	7.12×10^{-5}	1.02×10^{-4}
FI variance minimization	4.10×10^{-5}	6.21×10^{-6}	2.68×10^{-5}	2.94×10^{-6}

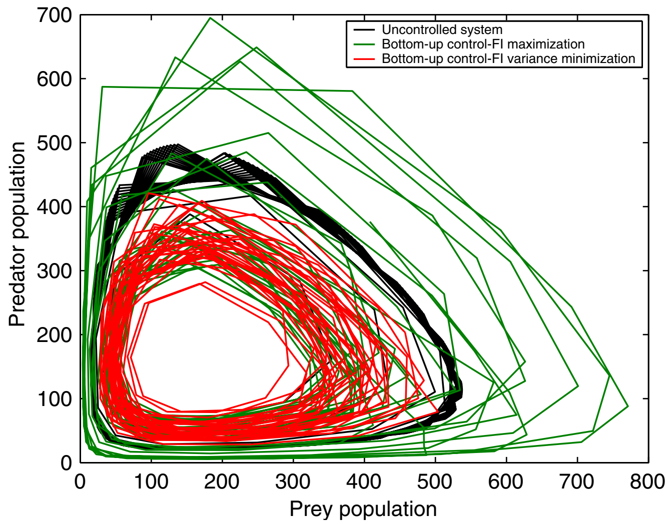

Fig. 3. Predator–prey population dynamics for case A: bottom-up control.

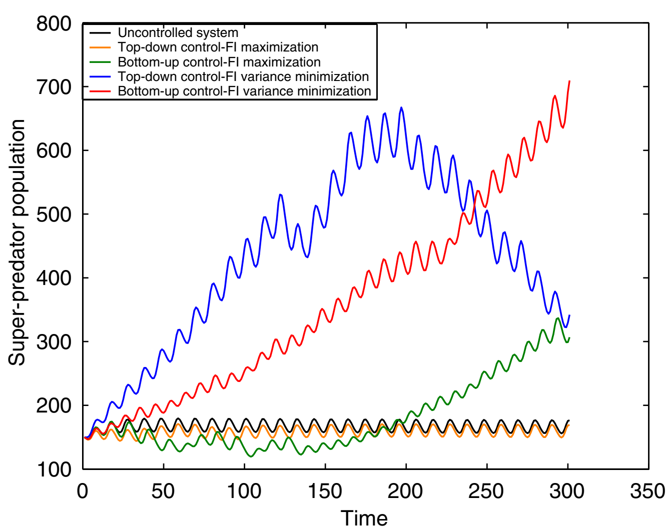

Fig. 4. Super-predator population dynamics for case A.

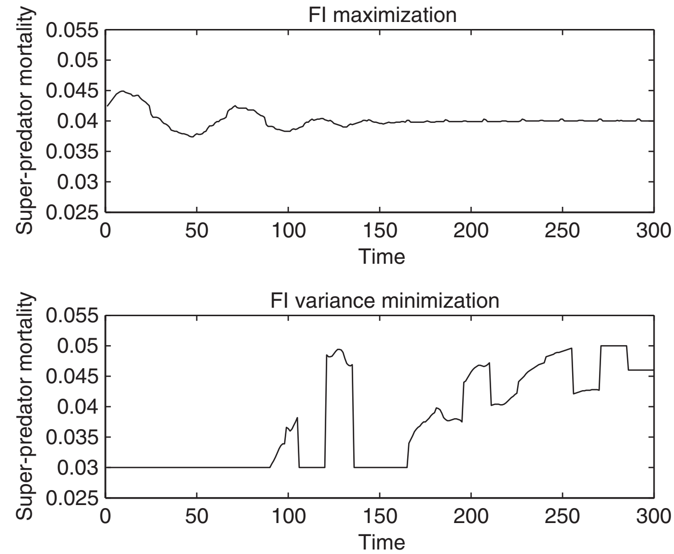

Fig. 5. Top-down control variable profile for case A.

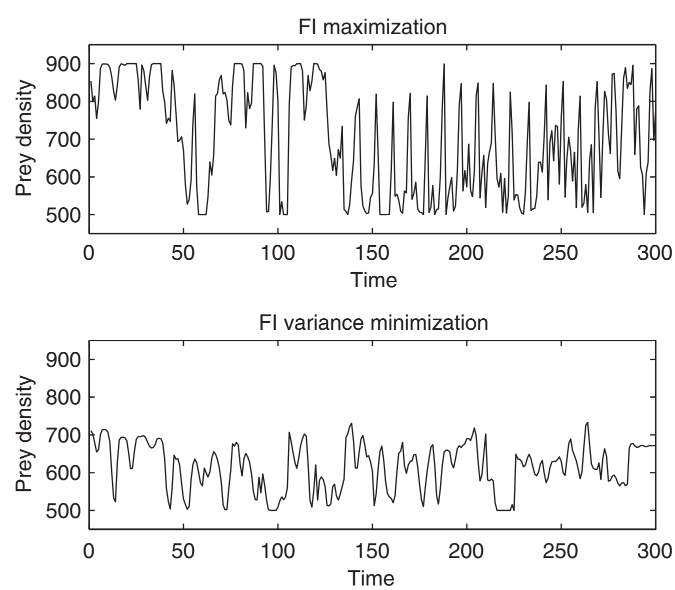

Fig. 6. Bottom-up control variable profile for case A.

that this control has very little effect on the prey and predator dynamics. Compared to this, the results for bottom-up control in Fig. 3 show a significant effect of the control action on the predator–prey population dynamics. The result for the FI variance minimization objective is desirable since it reduces the predator–prey variations, and achieves a dynamically stable system. Since the sustainability hypotheses argue that dynamic or static stability is the necessary condition for sustainability, the controlled system is also sustainable. The super-predator dynamics for these cases are plotted in Fig. 4. The favorable result by using bottom-up control for the FI variance minimization objective is accompanied by an increase in the super-predator population. However, a substantial rise in the super-predator population for the FI variance minimization objective using top-down control does not significantly affect the predator–prey dynamics. This observation indicates that the variations in the super-predator population are the effect and not the cause of reducing the predator–prey population variations. The control variable profiles for this case are given in Figs. 5 and 6. They show that the bottom-up control variable (K) fluctuates slightly more rapidly (at a higher frequency) for the FI maximization objective than for the FI variance minimization objective. The results for the FI variance minimization objective will, therefore, be easier to implement on an actual system, since the changes will be less frequent.

4.2. Case B: super-predator extinction

In order to simulate super-predator extinction, the value of b_2 (predator half saturation constant) is modified to

Table 4
Results for case B: super-predator extinction

Type of analysis	Top-down control		Bottom-up control	
	FI	FI standard deviation	FI	FI standard deviation
Uncontrolled model	4.71×10^{-5}	1.30×10^{-5}	4.71×10^{-5}	1.30×10^{-5}
FI maximization	4.72×10^{-5}	1.50×10^{-5}	8.30×10^{-5}	6.27×10^{-5}
FI variance minimization	4.72×10^{-5}	1.29×10^{-5}	3.44×10^{-5}	8.71×10^{-6}

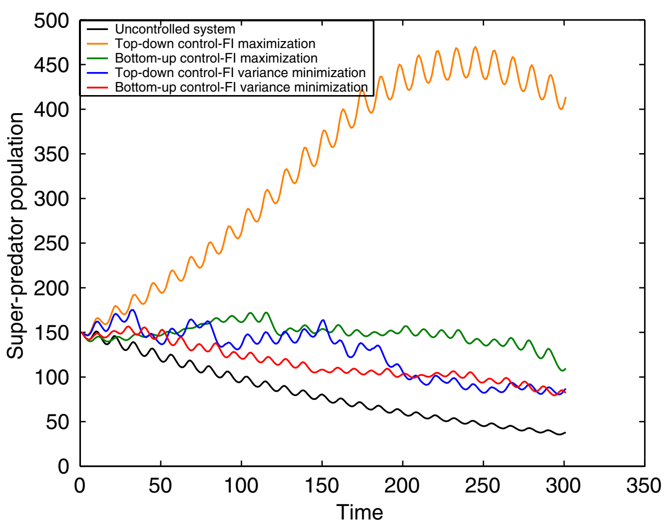

Fig. 7. Super-predator population dynamics for case B.

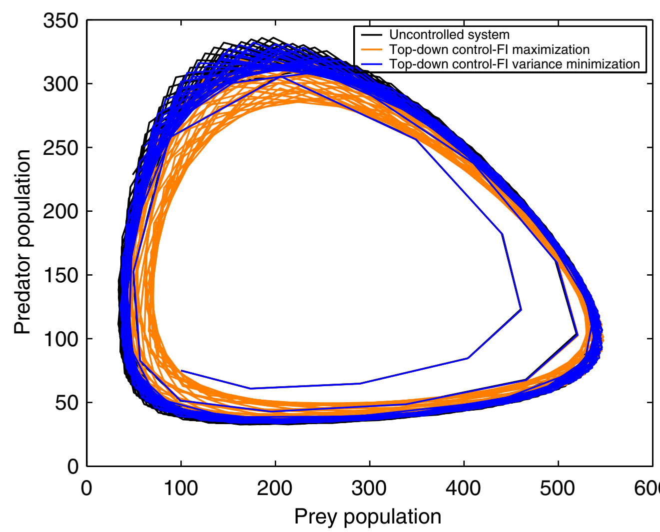

Fig. 8. Predator–prey population dynamics for case B: top-down control.

181.81. All the other parameter values are taken from Table 1. The numerical values of the results obtained for this case are reported in Table 4. The super-predator dynamics for this case are shown in Fig. 7. The plots show that all the control options are able to elevate the super-predator population above that for the uncontrolled case. The degree of success in restricting the super-predator extinction varies for different control options. The objective of FI maximization, for both control options, exerts a stronger impact, and for top-down control, the super-predator achieves a much higher population. The objective of FI variance minimization, on the other hand, shows a slower and weaker impact. The simulations of the system for a longer time duration show that the FI variance minimization objective is not able to elevate the super-predator population back to the initial level, but manages to avoid the super-predator extinction. Since the species extinction is avoided in all cases, they represent more sustainable systems as compared to the uncontrolled one. The predator–prey dynamics are shown in Fig. 8 (top-down control) and Fig. 9 (bottom-up control). It can be noticed that the effect of bottom-up control on the predator–prey population dynamics is stronger than the effect of top-down control. The control variable profiles for both the objectives are plotted in Fig. 10 (top-down control) and Fig. 11 (bottom-up control). As expected,

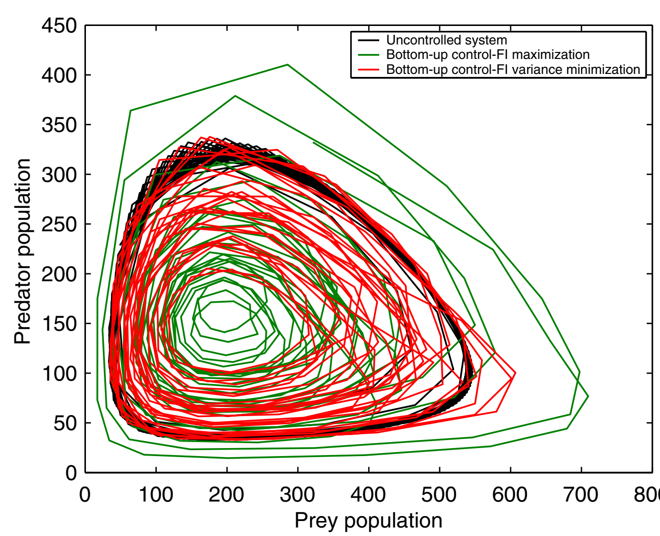

Fig. 9. Predator–prey population dynamics for case B: bottom-up control.

elevating the super-predator population increases the FI value. For top-down control, this is achieved by lowering the mortality rate to the minimum possible value (0.03) for about the first half of the time period. The higher than desired increase in the super-predator population is then compensated by a progressive increase in the mortality

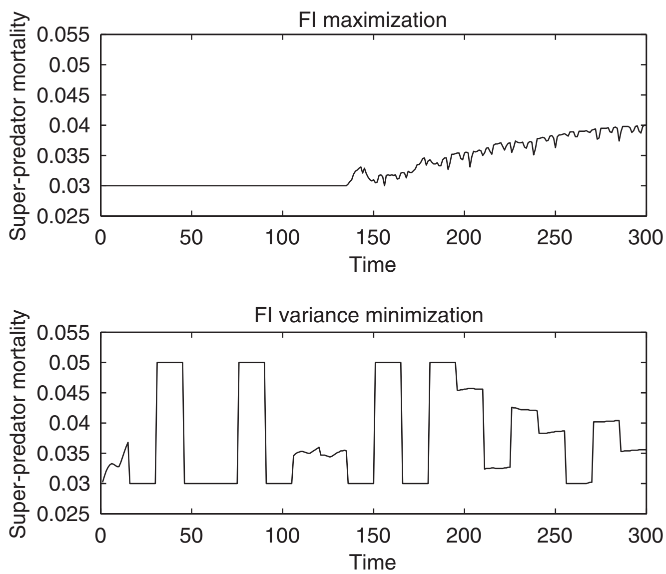

Fig. 10. Top-down control variable profile of case B.

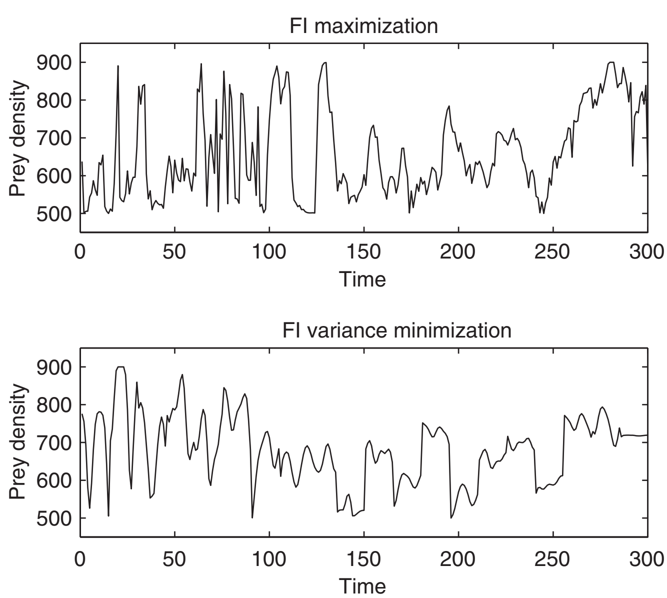

Fig. 11. Bottom-up control variable profile of case B.

rate. Since the FI variance minimization objective minimizes the variation between the average FI of each model cycle, the piecewise nature of the control action is evident in Fig. 10. The control actions are taken based on the dynamics of each cycle, and they keep fluctuating between the two extremes. The effect is evident in the super-predator population plot in Fig. 7, where the population is maintained around the starting point for the most part. The plots for bottom-up control also show the piecewise nature to a certain extent while using the FI variance minimization objective. The interpretation of these profiles is though not straightforward, since the effect on the super-predator population is indirect and nonlinear. The plots also indicate that the control variable profiles for the objective of FI maximization fluctuate a little more rapidly, and hence might be difficult to implement on a physical system. Another interesting observation from these results is the nonlinearity between the FI value and the associated population dynamics for the model. For the case of FI variance minimization objective using bottom-up control, the super-predator population increases and the prey and predators exhibit smaller population cycles. However, the average FI for this case is lower than the uncontrolled case, which clearly has the less desirable population dynamics.

4.3. Case C: super-predator explosion

The super-predator explosion is simulated by changing b_2 (predator half saturation constant) to 204.08, while the other parameters have the values reported in Table 1. The numerical values associated with the results obtained for this case are reported in Table 5. The super-predator dynamics are plotted in Fig. 12. It can be observed that the objective of FI variance minimization manages to restrict the super-predator population explosion. However, the objective of FI maximization does not give the desired results. Top-down control for this objective further supports the super-predator population explosion. This is because the relationship between the FI value and the super-predator population is nonlinear, as mentioned in the discussion of the case B results. Thus, for the FI variance minimization objective, the results, which restrict the super-predator population, have smaller average FI than the uncontrolled case. These results highlight the nonlinearity between the average FI value and the population dynamics, and suggest that the objective of FI maximization may not always lead to a desirable dynamic state of the system.

The predator–prey dynamics for this system are shown in Fig. 13 (top-down control) and Fig. 14 (bottom-up control). The plots reveal that the predator–prey dynamics are more significantly affected by bottom-up control, and this effect is more pronounced for the objective of FI maximization. The control variable profiles for both the objectives are shown in Fig. 15 (top-down control) and Fig.16 (bottom-up control). These plots again show the piecewise nature of the control variable when the FI variance minimization objective is used. In this case though, the difference in the nature of the control variable profiles for the two objectives is not as significant as in the previous cases, and hence, no specific comment about the ease of implementation can be made.

5. Regime change analysis

Drastic changes in the parameters of an ecosystem can cause shifts in the dynamic regimes of these systems, which can be stable or unstable. Such changes are caused by the natural disasters such as floods, hurricanes, or dramatic changes in the global climatic conditions (Mayer and Rietkerk, 2004). Quite often, these regime shifts are

Table 5
Results for case C: super-predator explosion

Type of analysis	Top-down control		Bottom-up control	
	FI	FI standard deviation	FI	FI standard deviation
Uncontrolled model	3.14×10^{-5}	7.76×10^{-6}	3.14×10^{-5}	7.76×10^{-6}
FI maximization	3.31×10^{-5}	8.89×10^{-6}	3.50×10^{-5}	1.09×10^{-5}
FI variance minimization	3.07×10^{-5}	6.21×10^{-6}	3.00×10^{-5}	3.92×10^{-6}

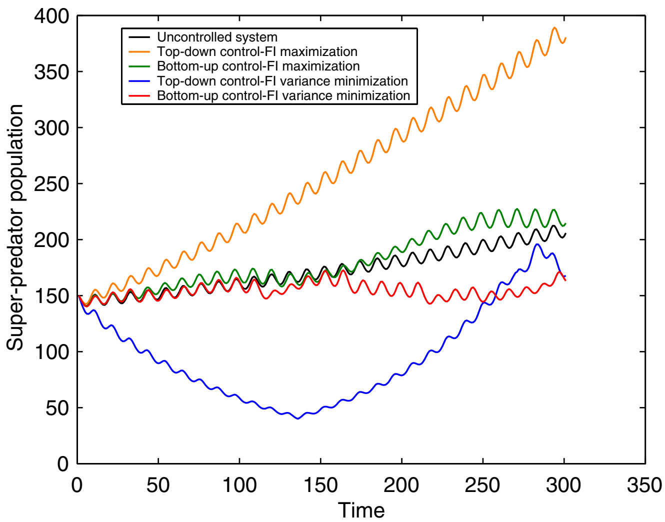

Fig. 12. Super-predator population dynamics for case C.

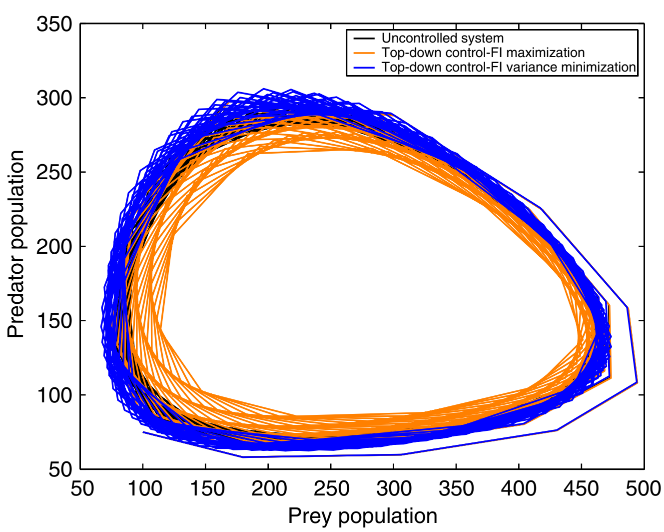

Fig. 13. Predator–prey population dynamics for case C: top-down control.

nonlinear, exhibiting phenomenon like hysteresis, meaning that the restoration of the original regime is complicated. Using time dependent manipulations of the system parameters, as discussed in this work, is a possible option to carry out this task.

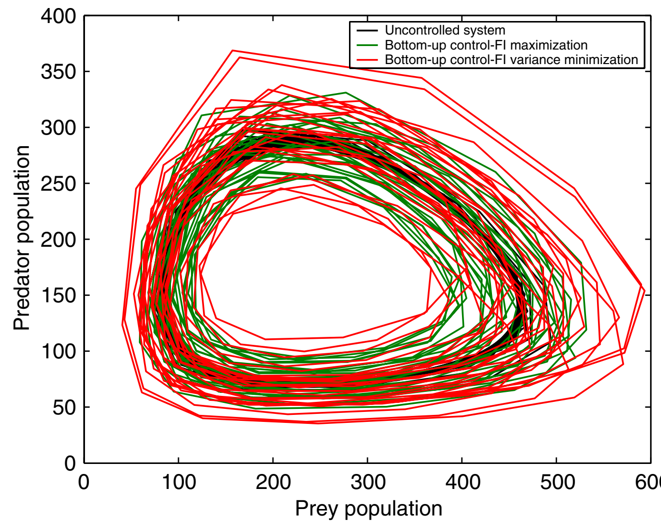

Fig. 14. Predator–prey population dynamics for case C: bottom-up control.

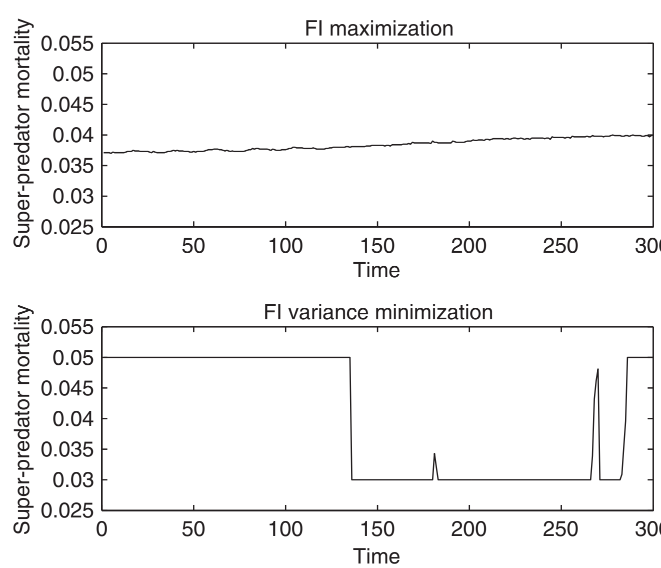

Fig. 15. Top-down control variable profile of case C.

The results presented in this section compare the top-down and bottom-up control philosophies for the tri-trophic food chain model, with the aim of shifting the population model from one dynamic regime (undesirable) to another (desirable). The same tri-trophic food chain

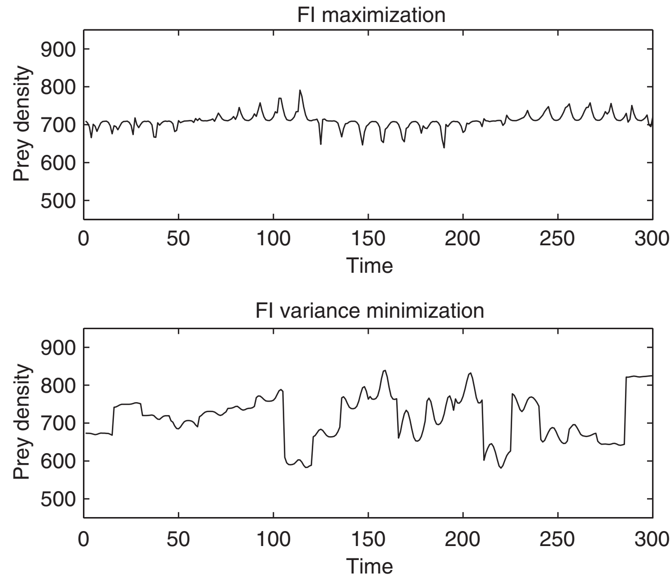

Fig. 16. Bottom-up control variable profile of case C.

Table 6
Tri-trophic food chain model: parameter set for the regime change analysis

Prey	Predator	Super-predator
$x_1(0) = 0.75$	$x_2(0) = 0.2$	$x_3(0) = 10$
$r = 1.5$	$a_2 = 1.67$	$a_3 = 0.05$
$K = 1.0$	$b_2 = 0.333$	$b_3 = 0.5$
	$e_2 = 1.0$	$e_3 = 1.0$
	$d_2 = 0.4$	$d_3 = 0.01$

model for a different set of parameters, reported in Table 6, is used for this analysis. The model parameters and the different regimes for this model are discussed in Gragnani et al. (1998). The possible regimes include the cyclic-low frequency regime and the cyclic-high frequency regime. The parameter set reported in Table 6 results in the cyclic-low frequency regime (an undesirable region, may be a result of a natural disaster). The population dynamics for this regime are shown in Fig. 17 (predator and prey populations) and Fig. 18 (super-predator population). The goal is to shift the system to the desirable cyclic-high frequency regime, which is also shown in Figs. 17 and 18. These regimes have different average FI values. The control philosophies try to shift the system regime by minimizing its FI variance around the average FI value of the desired regime (cyclic-high frequency). Thus, only the FI variance minimization objective is used. The super-predator mortality rate and the prey density are again the control variables for top-down and bottom-up control, respectively. The constraints on the control variables are: $0.005 \leqslant d_3 \leqslant 0.015$ for top-down control and $0.5 \leqslant K \leqslant 1.5$ for bottom-up control.

The results for the two control options are shown in Fig. 19 (predator and prey population) and Fig.20 (super-

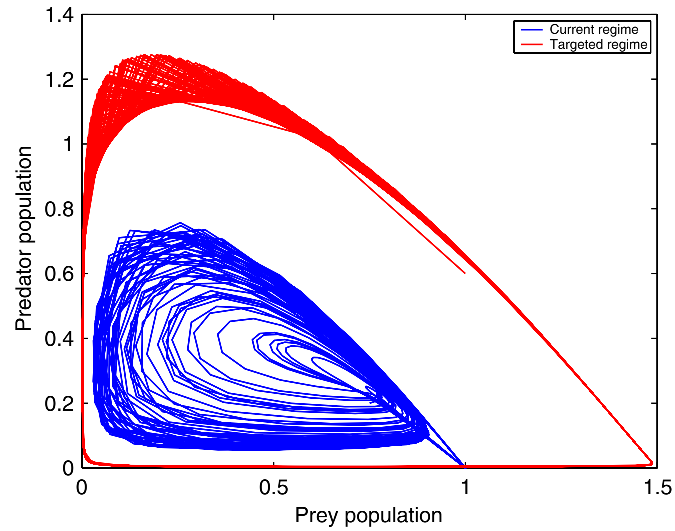

Fig. 17. Predator–prey population dynamics for different regimes in case D.

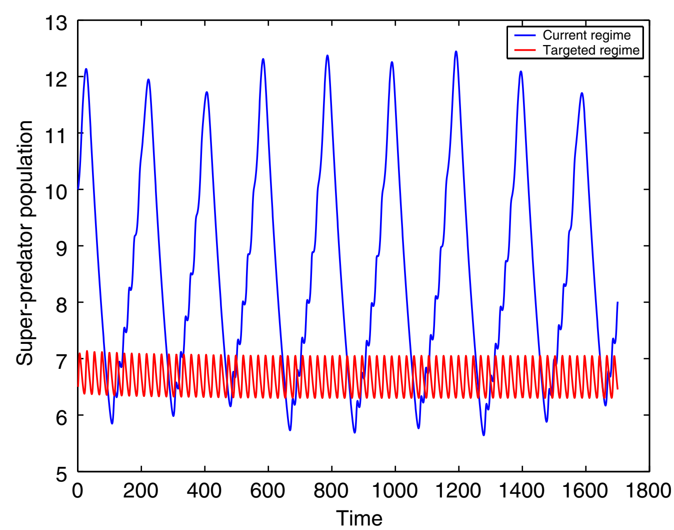

Fig. 18. Super-predator population dynamics for different regimes in case D.

predator population). It is clearly observed that the bottom-up controlled system shifts into the desired regime. On the contrary, the top-down controlled system fails to do so in the considered time horizon. This result illustrates that bottom-up control is a better option to affect regime changes, which might be required to recover a system from natural disasters such as hurricane Katrina.

6. Computational considerations

Since the system of equations being solved is quite complex, the computational problems need to be carefully avoided. Depending on the absolute value of FI, the objective function may need to be linearly scaled using a constant. This avoids the numerical errors during the solution of the equations. Experience shows that this

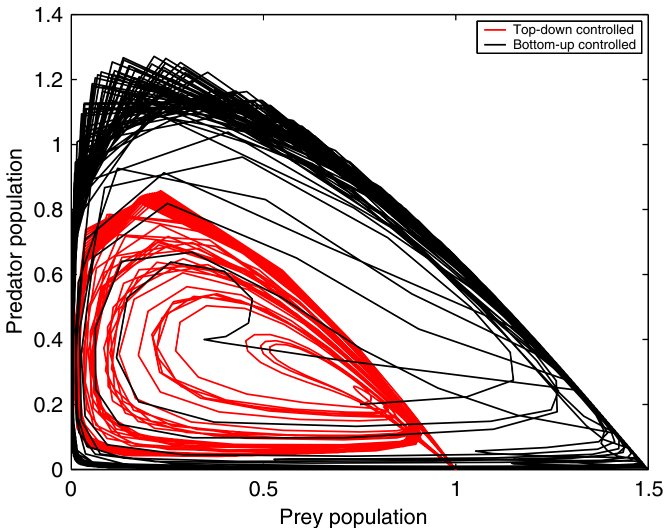

Fig. 19. Predator–prey population dynamics for controlled system in case D.

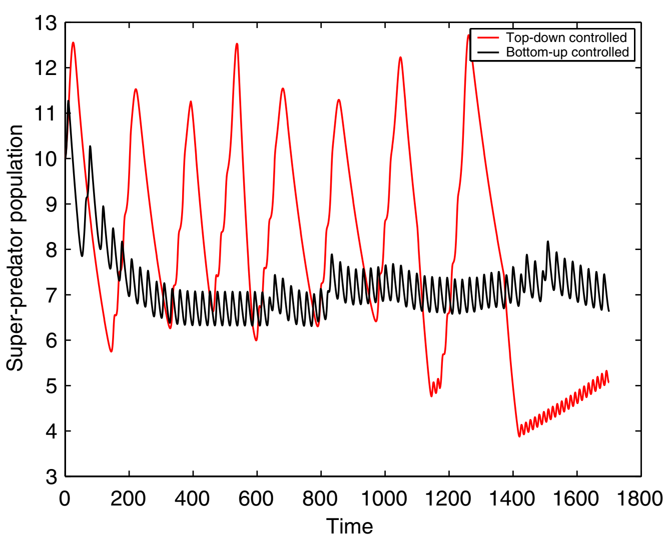

Fig. 20. Super-predator population dynamics for controlled system in case D.

greatly reduces the convergence time. The termination constant and the step size for the steepest ascent method need to be carefully chosen for converging results. Most often, it is a compromise between faster convergence and running the risk of making the solution divergent. It was also observed that a good initial guess is important for the convergence.

7. Conclusion

The objective of the work was to incorporate the ideas of sustainable development in the field of ecosystem management. For this, Fisher information(FI) along with the sustainability hypotheses was chosen as the indicator of sustainability. The predator–prey model, giving a simplistic representation of species interactions in the natural ecosystems, was considered for the application. Section 4 reported the results for the tri-trophic food chain model for different conditions. The objective was to compare two different control philosophies (top-down control and bottom-up control) and two different objectives, under different scenarios, and to assess their effect on the system dynamics. The definition of favorable system dynamics is quite subjective. Here, the system dynamics are considered to be favorable if the species show less population variations and/or if the species do not become extinct. The classification is based on the comparison with the uncontrolled system. The results show that favorable changes in FI, as suggested by the hypotheses, reflect, in general, in favorable system dynamics. This should give enough incentives for further investigations into the application of FI for ecosystem management. Following specific conclusions can be drawn from the results.

- The population dynamics and the FI value for a system are nonlinearly related, and hence, FI maximization is not guaranteed to improve the dynamics of the system. This is observed for case C, when the controlled system exhibits a further increase in the super-predator population over the uncontrolled system showing super-predator explosion. The super-predator population increases with an undiminished rate of increase for the considered time horizon. This suggests that the system might become unstable. Higher FI, therefore, does not necessarily represent a better dynamic system. The exact relationship might be system dependent.
- The objective of FI variance minimization achieves the goal of super-predator population control in cases B and C. This, along with the results in case A, suggests that the objective of FI variance minimization will not degrade the dynamics of a system.
- Bottom-up control has a greater impact on the predator–prey dynamics, evident by the populations dynamics considerably different from the uncontrolled case, and this impact is more significant for the FI maximization objective. Top-down control affects the super-predator population dynamics strongly, but its impact on the predator–prey dynamics is not significant.
- In terms of the absolute values of the objective functions, the bottom-up controlled systems obtain better values (i.e. higher time averaged FI and lower FI standard deviations) than the top-down controlled systems.
- The objective of FI maximization resulted in significantly worse values of the second objective (standard deviation), while the objective of FI variance minimization did not alter the value of the second objective (average FI) by much.
- The control variable profiles for the objective of FI variance minimization change somewhat less rapidly than for FI maximization. Since less frequent changes will be easier to implement on a physical system, the FI variance minimization objective might be preferable.

But the reported results are not conclusive enough to make a definite statement.

- Bottom-up control is more effective than top-down control in changing the dynamic regime of a system.

To summarize the results, one can argue that the objective of FI variance minimization will give a stable response without much disturbance, while the FI maximization objective is not guaranteed to improve the dynamics of a system. It may result in significant disturbances and increase the abundance of the superpredator. It was also observed that the FI variance minimization objective is not able to recover an uncontrolled system from significant disturbances (such as fast species extinction). Bottom-up control affects the populations of all species more significantly than top-down control. It is likely that some of these trends are system dependent. Based on the presented results though, FI variance minimization objective and bottom-up control option should be preferred in natural systems, where maintaining the system stability is more important than risking significant disturbance in an attempt to improve the state. On the contrary, for systems such as fisheries, where maximization of the fish harvesting is desirable, top-down control option and FI maximization objective should be preferred, particularly since the impacts can be kept localized and the system is under a better human control than completely natural systems. Bottom-up control, owing to its more widespread impact, should be chosen to affect major changes, such as regime shifts to recover from natural disasters.

The results presented here are based on the simulations of simplified models. It is well accepted that models can never truly represent the complex natural ecosystems. One might therefore doubt the validity of these findings. However, the experimental results also come with their own set of deficiencies. The results are quite often system specific (i.e. valid only for the particular lake or forest), nonreproducible, and the observations could be affected by many unknown factors not under direct human control. In such a situation, a strategic combination of the theoretical and the experimental approaches is needed. Theory helps in importing the novel and proven ideas from other fields into the field of ecosystem management, while experiments help in estimating the goodness of these findings. The results presented here are to be viewed with this perspective. The application of control theory to achieve sustainable ecosystems should guide an experimentalist to try different management options. It is expected that such an approach, if replaces logic and heuristics, will simplify the task of the experimental biologists.

Acknowledgements

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Appendix A

The equations used in the formulation of the optimal control equations in Section 3 are listed here. First defining parameters P_1 and P_2 as

$$P_1 = b_2 + x_1, \tag{A.1}$$

$$P_2 = b_3 + x_2. \tag{A.2}$$

A.1. Derivatives with respect to the control variable

A.1.1. d_3 as the control variable

$$\frac{\partial f_1}{\partial d_3} = 0, \tag{A.3}$$

$$\frac{\partial f_2}{\partial d_3} = 0, \tag{A.4}$$

$$\frac{\partial f_3}{\partial d_3} = -x_3, \tag{A.5}$$

$$\frac{\partial \bar{f}_1}{\partial d_3} = 0, \tag{A.6}$$

$$\frac{\partial \bar{f}_2}{\partial d_3} = \frac{a_3 x_2 x_3}{b_3 + x_2}, \tag{A.7}$$

$$\frac{\partial \bar{f}_3}{\partial d_3} = -\frac{2 e_3 a_3 x_2 x_3}{b_3 + x_2} + 2 d_3 x_3. \tag{A.8}$$

A.1.2. K as the control variable

$$\frac{\partial f_1}{\partial K} = \frac{r x_1^2}{K^2}, \tag{A.9}$$

$$\frac{\partial f_2}{\partial K} = 0, \tag{A.10}$$

$$\frac{\partial f_3}{\partial K} = 0, \tag{A.11}$$

$$\frac{\partial \bar{f}_1}{\partial K} = r^2 \frac{x_1^2}{K^2}\left(3 - 4\frac{x_1}{K}\right) - \frac{r a_2 x_1^2 x_2}{K^2 P_1}\left(2 - \frac{b_2}{P_1}\right), \tag{A.12}$$

$$\frac{\partial \bar{f}_2}{\partial K} = \frac{r e_2 a_2 x_1^2 x_2 b_2}{K^2 P_1^2}, \tag{A.13}$$

$$\frac{\partial \bar{f}_3}{\partial K} = 0. \tag{A.14}$$

A.2. Derivatives with respect to the state variables

A.2.1. State variable x_1

$$\frac{\partial f_1}{\partial x_1} = r - \frac{2rx_1}{K} - \frac{a_2 x_2 b_2}{P_1^2}, \tag{A.15}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{a_2 e_2 b_2 x_2}{P_1^2}, \tag{A.16}$$

$$\frac{\partial f_3}{\partial x_1} = 0, \tag{A.17}$$

$$\begin{aligned}\frac{\partial \bar{f_1}}{\partial x_1} = {} & \left[r\left(1 - \frac{x_1}{K}\right) - a_2\frac{x_2}{P_1}\right]\left[r - 4r\frac{x_1}{K} - \frac{a_2 x_2 b_2 (b_2 - x_1)}{P_1^3}\right] \\ & + \left[\frac{-r}{K} + a_2\frac{x_2}{P_1^2}\right]\left[rx_1 - \frac{2rx_1^2}{K} - \frac{a_2 b_2 x_1 x_2}{P_1^2}\right] \\ & - \frac{a_2 b_2 x_2}{P_1^2}\left[2e_2 a_2\frac{x_1}{P_1} - a_3\frac{x_3}{P_2} - d_2\right],\end{aligned} \tag{A.18}$$

$$\begin{aligned}\frac{\partial \bar{f_2}}{\partial x_1} = {} & \frac{e_2 a_2 b_2 x_2}{P_1^2}\left[2e_2 a_2\frac{x_1}{P_1} - a_3\frac{x_3}{P_2} - 2d_2\right] \\ & + \frac{e_2 a_2 b_2 x_1 x_2}{P_1^2}\left[-\frac{r}{K} + a_2\frac{x_2}{P_1^2}\right] \\ & + \left[r\left(1 - \frac{x_1}{K}\right) - a_2\frac{x_2}{P_1}\right]\frac{e_2 a_2 b_2 x_2 (b_2 - x_1)}{P_1^3} \\ & - \frac{e_2 a_2 a_3 b_2 b_3 x_2 x_3}{P_1^2 P_2^2},\end{aligned} \tag{A.19}$$

$$\frac{\partial \bar{f_3}}{\partial x_1} = \frac{e_2 e_3 a_2 a_3 b_2 b_3 x_2 x_3}{P_1^2 P_2^2}. \tag{A.20}$$

A.2.2. State variable x_2

$$\frac{\partial f_1}{\partial x_2} = -a_2\frac{x_1}{P_1}, \tag{A.21}$$

$$\frac{\partial f_2}{\partial x_2} = \frac{a_2 e_2 x_1}{P_1} - \frac{a_3 b_3 x_3}{P_2^2} - d_2, \tag{A.22}$$

$$\frac{\partial f_3}{\partial x_2} = \frac{e_3 a_3 b_3 x_3}{P_2^2}, \tag{A.23}$$

$$\begin{aligned}\frac{\partial \bar{f_1}}{\partial x_2} = {} & -\frac{a_2 x_1}{P_1}\left[e_2 a_2\frac{x_1}{P_1} - a_3\frac{x_3}{P_2} - d_2\right] \\ & - \frac{a_2 b_2 x_1}{P_1^2}\left[r\left(1 - \frac{x_1}{K}\right) - 2a_2\frac{x_2}{P_1}\right] \\ & - \frac{r a_2 x_1}{P_1}\left(1 - \frac{2x_1}{K}\right) - \frac{a_2 a_3 x_1 x_2 x_3}{P_1 P_2^2},\end{aligned} \tag{A.24}$$

$$\begin{aligned}\frac{\partial \bar{f_2}}{\partial x_2} = {} & \left[e_2 a_2\frac{x_1}{P_1} - a_3\frac{x_3}{P_2} - d_2\right] \\ & \times\left[\frac{e_2 a_2 x_1}{P_1} + \frac{e_3 a_3^2 b_3 x_2 x_3}{P_2^3} - \frac{d_2 a_3 x_2 x_3}{P_2^2}\right] \\ & + \frac{e_2 a_2 b_2 x_1}{P_1^2}\left[r\left(1 - \frac{x_1}{K}\right) - 2a_2\frac{x_2}{P_1}\right] \\ & - \frac{a_3 b_3 x_3}{P_2^2}\left[e_3 a_3\frac{x_2}{P_2} - d_3\right] + \frac{a_3 x_2 x_3}{P_2^2} \\ & \times\left[e_2 a_2\frac{x_1}{P_1} - \frac{a_3 b_3 x_3}{P_2^2} - \frac{e_3 a_3 b_3}{P_2} - d_2\right],\end{aligned} \tag{A.25}$$

$$\begin{aligned}\frac{\partial \bar{f_3}}{\partial x_2} = {} & \frac{e_3 a_3 b_3 x_3}{P_2^2}\left[e_3 a_3\frac{x_2}{P_2} - 2d_3\right] \\ & + \frac{e_3 a_3^2 b_3 x_2 x_3}{P_2^3}\left(e_3 - \frac{x_3}{P_2}\right) \\ & + \left[e_2 a_2\frac{x_1}{P_1} - a_3\frac{x_3}{P_2} - d_2\right]\frac{e_3 a_3 x_3 b_3 (b_3 - x_2)}{P_2^3}.\end{aligned} \tag{A.26}$$

A.2.3. State variable x_3

$$\frac{\partial f_1}{\partial x_3} = 0, \tag{A.27}$$

$$\frac{\partial f_2}{\partial x_3} = -\frac{a_3 x_2}{P_2}, \tag{A.28}$$

$$\frac{\partial f_3}{\partial x_3} = \frac{e_3 a_3 x_2}{P_2} - d_3, \tag{A.29}$$

$$\frac{\partial \bar{f_1}}{\partial x_3} = \frac{a_2}{P_1} x_1 x_2 \frac{a_3}{P_2}, \tag{A.30}$$

$$\begin{aligned}\frac{\partial \bar{f_2}}{\partial x_3} = {} & -\frac{a_3 b_3 x_2}{P_2^2}\left[e_2 a_2\frac{x_1}{P_1} - 2a_3\frac{x_3}{P_2} - d_2\right] \\ & - \frac{a_3 x_3}{P_2}\left[\frac{e_3 a_3 x_2}{P_2} - d_2 - d_3\right] - \frac{e_2 a_2 a_3 x_1 x_2}{P_1 P_2},\end{aligned} \tag{A.31}$$

$$\begin{aligned}\frac{\partial \bar{f_3}}{\partial x_3} = {} & \left[e_3 a_3\frac{x_2}{P_2} - d_3\right]^2 \\ & + \frac{e_3 a_3 b_3 x_2}{P_2^2}\left[e_2 a_2\frac{x_1}{P_1} - 2a_3\frac{x_3}{P_2} - d_2\right].\end{aligned} \tag{A.32}$$

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